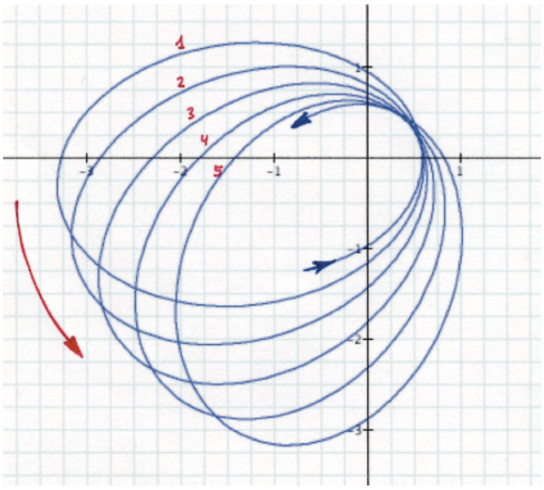
The Spiral Galaxy M66 (NGC 3627)
VLT MELIPAL/YEPUN + FORS1/FORS2

ESO PR Photo 33c/03 (19 December 2003)

I Testing the General Theory of Relativity

In this section we present both old and recent observations and experiments which Newton's theory can not explain but which the GTR predicts to a great precision. Some of these experiments, such as the deflection of light at the solar limb, the time delay from Shapiro or the direct measurements of the effects of time using atomic clocks, can be calculated with our current resources to a good approximation of the expected results. For the perihelion of Mercury we can at least estimate the magnitude of the effect. For other experiments we must be satisfied that at least we understand that an effect is expected.

I1 The Precession of the Perihelion of Mercury



Should a single, isolated planet orbit the sun then it must do so according to Kepler and Newton in an exact ellipse. Newton already recognized that this is not the case in the solar system because the planets affect each other gravitationally. An exact solution to the 'three-body problem' evaded even great people like Poincaré (whose attempt at a solution deeply penetrated into the territory known today as 'chaos theory'). Today, iterative numeric methods can calculate the orbits of all the planets with high precision for a long time into the future. The apsis, the straight line through the aphelion (furthest point from sun) and perihelion (closest point to sun) of the orbit precedes very slowly under the influence of the outer planets, in the same direction in which the planets orbit. This results in a rosette-like path, where the effect as shown in the diagram is greatly exaggerated. By the way, these numerical simulations have also shown that the solar system will remain stable even over very long periods [36-315ff].

There is, however, a small difference between the calculated values for the precession of the perihelion within Newtonian physics and those measured observationally. The following table shows the numerical values in the units 'arc seconds per century'. The fuzziness of the values can be read from the column 'difference':

Planet	Newtonian value	Observed Value	Difference	Prediction GTR
Merkur	532.08	575.19	43.11 ± 0.45	43.03
Venus	13.2	21.6	8.4 ± 4.8	8.6
Erde	1165	1170	5 ± 1.2	3.8

The difference between the calculated and the measured value, especially in the case of Mercury was so big that it demanded an explanation. The French astronomer Urbain Le Verrier, who predicted in 1845 the existence and location of the new planet Neptune based on the interference of the planet Uranus, postulated in 1859 the existence of another planet Vulcan, whose orbit was closer to the sun than Mercury's.

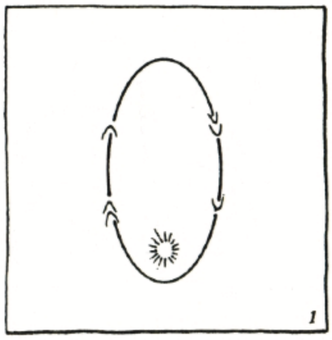
The GTR explains precisely this difference between the Newtonian theory and observation. Einstein was overjoyed when he calculated near the end of 1915 that his new theory predicted an addition of just 43 arc seconds per century to the precession of the perihelion of Mercury! He derived the following formula:

$$\Delta\varphi = 3 \cdot \pi \cdot \frac{R_s}{a \cdot (1-\varepsilon^2)}$$

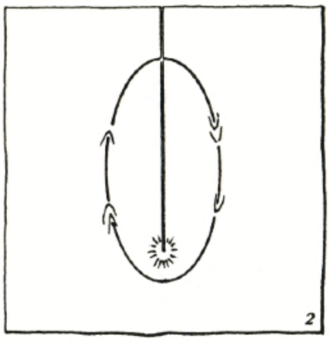
where $\Delta\phi$ is the extra rotation per orbit in radians; R_S is the Schwarzschild radius of the sun; a is the length of the semi-major axis of the orbit; and ε is the eccentricity of the ellipse.

The effect decreases with increasing distance from the sun and is also greater with highly elliptical orbits than with circular orbits. Therefore Mercury was an ideal candidate. The small eccentricity of the orbit of Venus not only weakens the effect but also makes it difficult to observe the precession. The values in the last column of the table can be obtained from Einstein's formula, if the result is multiplied by the number of revolutions in 100 years and then converted from radians to arc seconds (See problem 1).

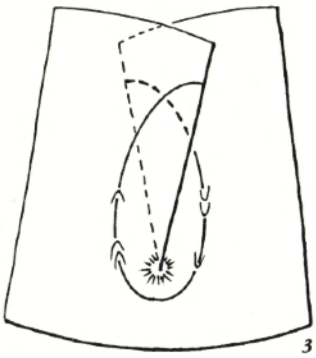
That this effect *must* occur is made nearly self-evident by Epstein's 'barrel' region [15-166]:



In the first drawing space is flat and the planet moves in its ellipse, however, with Epstein in an unconventional direction (usually one always looks from the north to the ecliptic). That is the situation according to Newton.

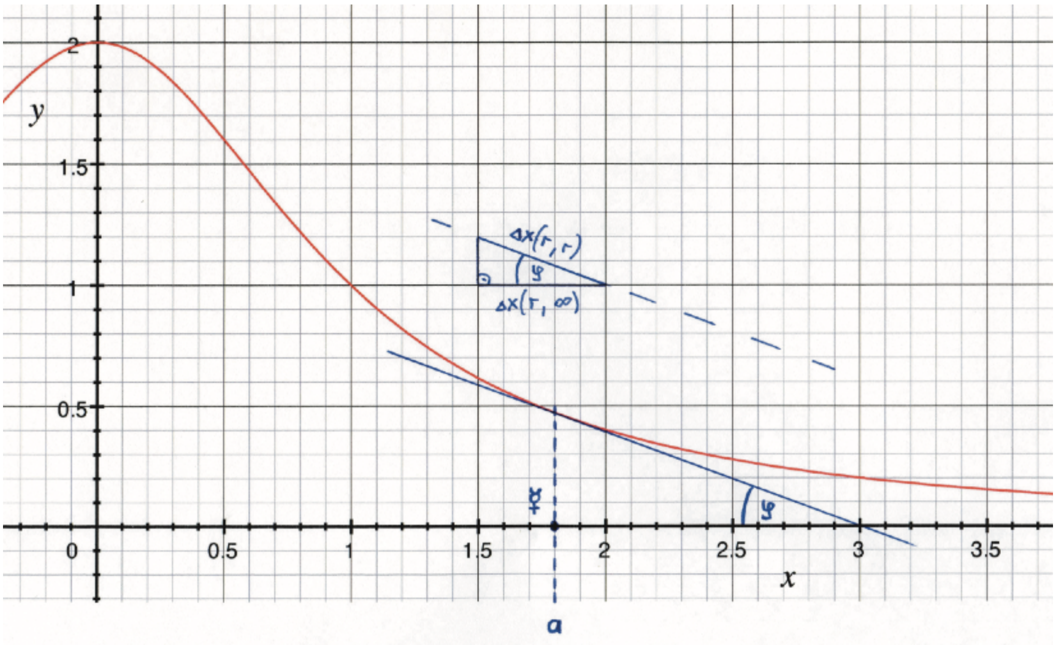


Now we cut the plane along the apsis. We make the cut from the aphelion to the sun.

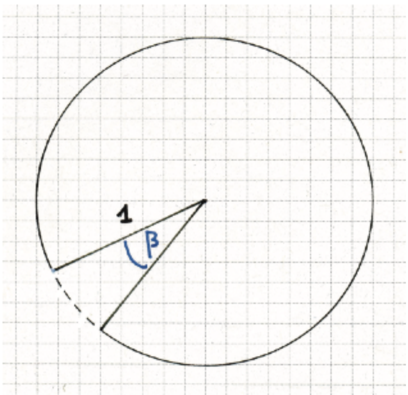


As discussed in section **H6** a cone should now arise with its tip in the sun. To this end we must push the edges on both sides of the incision over each other (thus crafts one a cone!). This forces an advance (precession) of the aphelion in the direction of the planet's orbit!

Amazingly, it is even possible to determine the magnitude of the phenomenon from Epstein's paper model. Almost with no computation we come surprisingly close to the results of the formula, whose derivation forced Einstein to tussle with elliptic integrals.



The red curve shows the cross-section of Epstein's 'space bump' (problem 5 in **I10** deals with the analysis of this function). The central mass sits at the origin, and with increasing distance x from the origin the spatial curvature decreases. If our planet has an average distance a from the central mass, we can then approximate the space bump with a local cone. The appropriate angle of inclination ϕ between the surface line of the cone and the plane through the center of the central mass can be easily determined for the planet at point a. It is $\cos(\phi) = \Delta x(r, \infty) / \Delta x(r, r) = 1 - \alpha / a$ according to **G4**. If the cone has a surface line of length 1 then the base circle measures a radius of $(1 - \alpha / a)$. We now cut the cone along a surface line and press it flat:



How big is the angle β of the missing sector?

$\beta / (2\pi)$ is equal to the ratio of the 'missing' arc length to the circumference, i.e.,
$\beta / (2\pi) = [2\pi - 2\pi \cdot (1 - \alpha / a)] / (2\pi) = 1 - (1 - \alpha / a) = \alpha / a$

We thus obtain the expression for β
$\beta = (2\pi) \cdot \alpha / a = \pi \cdot 2 \cdot \alpha / a = \pi \cdot R_S / a$

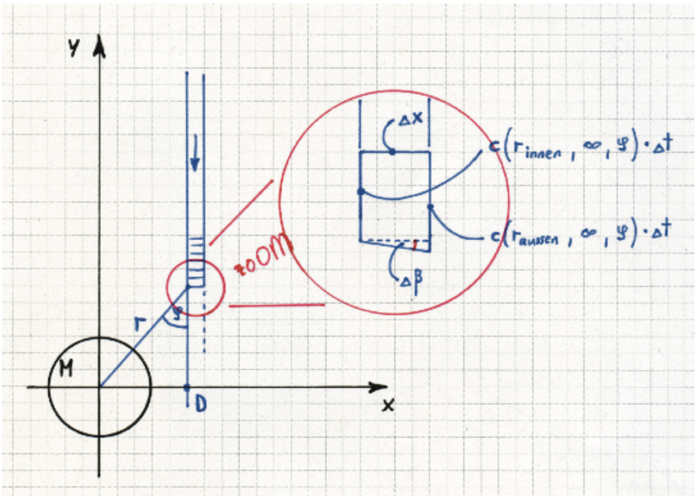
Reform the cone and you will find either a circle or an ellipse offset by about the size of this angle β, that is, β specifies the amount of precession per orbit of the apsis.

We only obtain about one third of the correct value (compare with the formula on p.134 above). This should not concern us since we have only taken the influence of space curvature into account, and that also using very modest means. We are, in any case, within a correct order of magnitude.

I2 The Deflection of Light in the Gravitational Field of the Sun

We now want to do the calculations for the experiment, whose outcome made Einstein so famous in 1919: The deflection of light at the solar limb. The effect on the public and the clamor of the press concerning this result can only be understood against the backdrop of the just ended disaster of the First World War (see [32-232ff] or [38-191ff]). In **H6**, we already stressed that the outcome of this experiment speaks in favor of the GTR and against Newton's theory which expected only half the value predicted by the GTR.

We take the following approach: a light beam of width Δx passes the sun at a distance D along the y-axis. Thereby, according to our formulas in **G5**, the inner side of the beam which is closer to the sun, moves by a little bit slower than the outer side, so the wave front is tilted by a small angle Δß:



To calculate the sum of all these small changes of direction Δß, we will integrate from $y = +\infty$ to $y = -\infty$ using the constant value D for x. In so doing, we require that the entire change in direction is small. In principle, this is 'gravitation by refraction'!

Step 1

We determine a workable expression for the infinitesimal change in direction Δß:

$$\Delta\beta \approx \tan(\Delta\beta) = \frac{\left[c(x+\Delta x,y,\infty,\varphi) - c(x,y,\infty,\varphi)\right]\cdot\Delta t}{\Delta x} = \frac{c(x+\Delta x,y,\infty,\varphi) - c(x,y,\infty,\varphi)}{\Delta x}\cdot\Delta t$$

We must integrate over the y-axis. We are not in error when we write $dy = c_0 \cdot dt$, even if the light beam (as seen from OFF) does not quite advance with velocity c_0. This really only means that our time slices dt are not all of the same size. Still we may overall write

$$d\beta \approx \frac{\partial(c(r,\infty,\varphi))}{\partial x}\cdot dt = \frac{\partial(c(r,\infty,\varphi))}{\partial x}\cdot\frac{1}{c_0}\cdot dy$$

So if we knew the partial derivative of the speed of light with respect to x, then we could write our integral:

$$\beta_{total} = \int_{+\infty}^{-\infty} d\beta\cdot dy = \int_{+\infty}^{-\infty}\frac{\partial(c(r,\infty,\varphi))}{\partial x}\cdot\frac{1}{c_0}\cdot dy$$

Step 2

We determine the partial derivative of c(r, ∞, φ) with respect to x. The calculation is a bit tedious:

$$\frac{\partial}{\partial x}[c(r,\infty,\varphi)] = \frac{\partial}{\partial x}\left[c_0\cdot\left(1-\frac{\alpha}{r}\cdot\left(1+\cos^2(\varphi)\right)\right)\right] =$$

$\cos(\varphi) = y/r$

$$c_0\cdot\frac{\partial}{\partial x}\left[1-\frac{\alpha}{r}-\frac{\alpha}{r}\cdot\cos^2(\varphi)\right] = c_0\cdot(-\alpha)\cdot\frac{\partial}{\partial x}\left[\frac{1}{r}+\frac{1}{r}\cdot\cos^2(\varphi)\right] =$$

$r^2 = x^2 + y^2$

$$c_0\cdot(-\alpha)\cdot\frac{\partial}{\partial x}\left[\frac{1}{r}+\frac{y^2}{r^3}\right] = c_0\cdot(-\alpha)\cdot\frac{\partial}{\partial x}\left[\frac{r^2+y^2}{r^3}\right] =$$

now we do derive

$$c_0\cdot(-\alpha)\cdot\frac{\partial}{\partial x}\left[\frac{x^2+y^2+y^2}{(x^2+y^2)^{3/2}}\right] = c_0\cdot(-\alpha)\cdot\frac{\partial}{\partial x}\left[\frac{x^2+2\cdot y^2}{(x^2+y^2)^{3/2}}\right] =$$

$$c_0\cdot(-\alpha)\cdot\frac{(x^2+y^2)^{3/2}\cdot 2x-(x^2+2\cdot y^2)\cdot\frac{3}{2}\cdot(x^2+y^2)^{1/2}\cdot 2x}{(x^2+y^2)^3} =$$

divide by $(x^2+y^2)^{0.5}$

$$c_0\cdot(-\alpha)\cdot\frac{(x^2+y^2)\cdot 2x-3\cdot(x^2+2\cdot y^2)\cdot x}{(x^2+y^2)^{2.5}} =$$

$$c_0\cdot(-\alpha)\cdot\frac{2x^3+2xy^2-3x^3-6xy^2}{(x^2+y^2)^{2.5}} = c_0\cdot(-\alpha)\cdot\frac{-x^3-4xy^2}{(x^2+y^2)^{2.5}} =$$

$$c_0\cdot\alpha\cdot\frac{x^3+4xy^2}{(x^2+y^2)^{2.5}} = c_0\cdot\alpha\cdot\frac{x^3+4xy^2}{r^5}$$

$r^2 = x^2 + y^2$

Step 3

We evaluate the integral:

$$\beta_{total} = \int_{+\infty}^{-\infty}\frac{\partial(c(r,\infty,\varphi))}{\partial x}\cdot\frac{1}{c_0}\cdot dy = \int_{+\infty}^{-\infty}c_0\cdot\alpha\cdot\frac{x^3+4xy^2}{(x^2+y^2)^{2.5}}\cdot\frac{1}{c_0}\cdot dy =$$
$$= \alpha\cdot\int_{+\infty}^{-\infty}\frac{x^3+4xy^2}{(x^2+y^2)^{2.5}}\cdot dy = \alpha\cdot\int_{+\infty}^{-\infty}\frac{D^3+4Dy^2}{(D^2+y^2)^{2.5}}\cdot dy$$

On the light's path D = x = 2.33 light seconds is constant. In place of +∞ to -∞ one could also integrate from 2000 to -2000 light seconds (the earth is about 500 light seconds from the sun). If so then note that α must also be in these units: $\alpha = G\cdot M/c^2 = G'\cdot M \approx 4.9261\cdot 10^{-6}$ light seconds !

Using the TI-89 calculator to do the integration delivers a result of $8.4571\cdot 10^{-6}$. This is the whole ß deflection in radians. Converting to arc seconds, we get $8.4571\cdot 10^{-6}\cdot 180\cdot 3600 / \pi \approx 1.74$ arc seconds. Using Mathematica ® with twice the precision gives the same value, namely 1.7518 arc seconds.

For our integral, [27-143] gives the anti-derivative $(y / r + (y / r)^3) / D$. This can easily be checked by differentiating! Now the limit of y / r as y approaches infinity is simply 1. Thus for the total deflection in radians one obtains the simple formula

$$\boxed{\beta_{total} = \frac{4\cdot\alpha}{D} = \frac{4\cdot G\cdot M}{c^2\cdot D} = \frac{2\cdot R_s}{D}}$$

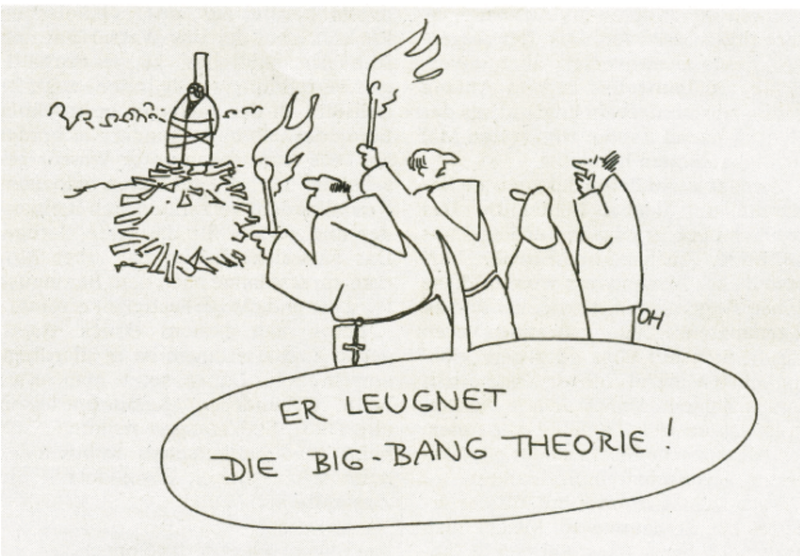
What experimental data are available to test this formula?

[27-145] presents the analysis of photographs of star fields during solar eclipses up till 1952. The values lie between 1.61" and 2.01" with uncertainties of 0.10” to 0.45” (I have omitted so-called ‘outliers’). This is enough to give GTR preference over Newton's theory, but the uncertainty is greater than 10%. The ‘phone book’ [29-1105] gives data from measurements with radio telescopes. Every year on October 8, the sun – as seen from the earth - moves over the quasar 3C279. Another quasar 3C273 is in the vicinity and allows a precise measurement of the angle between these two objects. In 1970 the GTR could be confirmed to within 5% accuracy. Measurements using VLBI (very long baseline interferometry) could confirm GTR in 1995 to an accuracy of 0.9996 ± 0.0017, i.e., to 1.7 parts per thousand. In 1999 an analysis of 2 million VLBI measurements was published, which delivered an accuracy of 0.99992 ± 0.00014. This data were taken from the following website on December 26, 2006:

http://relativity.livingreviews.org/open?pubNo=lrr-2001-4&page=node10.html

The page **http://relativity.livingreviews.org** due mainly to Clifford M. Will has rendered outstanding service and provides the most comprehensive and current information about ongoing research and experimental trials in the field of GTR.

Another confirmation of GTR comes from the data of ESO's Hipparcos satellite. This had the task of precisely measuring the position of 118,000 stars (up to size class 12.5), so that they could later be used as reference. Hipparcos (after a very inauspicious start, see Wikipedia) performed this task brilliantly. The angular resolution of the position measuring instruments was 0.001 arc seconds. Thus, it could confirm the predictions of GTR across the entire sky with an accuracy of about 0.3%. Light deflection can already be measured, if the light moves past the sun at a distance of one astronomical unit, i.e., a distance of 150 million kilometers! When we direct our sight from the earth along the y-axis, i.e., in a direction perpendicular to the segment connecting the earth to sun, then the light beam has already suffered exactly half of the total ß deflection. $ß = 2 \cdot R_S / D$ with D = 1 AU has the value 0.0081 arc seconds, and half is still 4 thousandths of an arc second, i.e., four times the accuracy of the Hipparcos satellite!

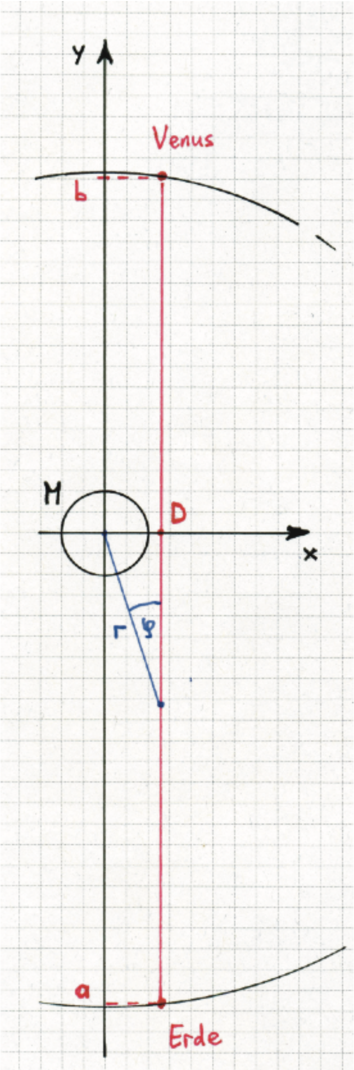


"He denies the Big bang Theory !"

Oswald Huber, Neue Zürcher Zeitung, Sunday Edition, November 12, 2006

I3 The Shapiro Experiment

In **I2** we saw that a beam of light that passes close to the sun barely changes its direction. On the other hand, the time needed for the traversal of a given path diameter grows significantly, because the light, in accordance with the formulas of **G5**, moves more slowly than it would when outside a gravitational field. Problem 5 of **H7** illustrates the effect of spatial curvature, but a virtually equal proportion is also due to the curvature of space-time. It is again an effect that can be well understood viewed as 'gravitation by refraction'.



In 1962 Irwin Shapiro suggested that this delay be measured by sending some strong radio signals to Venus, when it is in opposition to the Earth, and then measuring the time it takes for the (extremely weak) reflected signals to arrive.

When in 1964 the 120 foot Haystack antenna in Westford, US was left by the military to MIT, Shapiro and his team began plans to carry out the experiment. The experiment first took place from November 1966 until August 1967. "It would have been nice to prove Einstein wrong," said Shapiro later. That has not been granted him since all experiments up till 2006 have confirmed the GTR within the specified accuracy.

Shapiro lowered the imprecision of his initial measurements from over 3% to less than 1% in subsequent years. Newer versions of this experiment work with transponders on space probes. These receive the signal from the earth and after a precisely known delay send it with increased intensity back to earth. Thus with the Viking Mars probe of 1979 the predictions of the GTR for this delay in the gravitational field of the sun could be confirmed to an accuracy of 0.1%. In 2003, with the space probe Cassini an accuracy of 0.0012% was achieved!

In the situation presented on the left an observer in OFF would measure values of a = -498.67 and b = 370.70 (as in **I2**, we calculate everything in units of light seconds, so that $c_0 = 1$ and $\alpha \approx 4.9261 \cdot 10^{-6}$). Without gravity one would expect a duration of $2 \cdot (b - a) / c_0 \approx 2 \cdot (370.70 + 498.67) / 1 \approx 1738.74$ seconds. In the following we calculate the difference in time that arises because the light near the sun travels a little bit slower.

With gravity the duration in both directions (with $c_0 = 1$!) is

$$T = 2 \cdot \int_a^b \frac{1}{c(r,\infty,\varphi)} \cdot dy = 2 \cdot \int_a^b \frac{1}{c_0 \cdot \left(1 - \left(1 + \cos^2(\varphi)\right) \cdot \alpha / r\right)} \cdot dy = 2 \cdot \int_a^b \frac{1}{1 - \left(1 + \cos^2(\varphi)\right) \cdot \alpha / r} \cdot dy$$

This integral is numerically very unstable. A substitution using $1/(1-x) = (1+x)/(1-x^2)$ helps since we then eliminate in the denominator the very small x^2 term:

$$T = 2\cdot\int_a^b \frac{1+\left(1+\cos^2(\varphi)\right)\cdot\alpha/r}{1-\left(\left(1+\cos^2(\varphi)\right)\cdot\alpha/r\right)^2}\cdot dy \approx 2\cdot\int_a^b 1+\left(1+\cos^2(\varphi)\right)\cdot\alpha/r\cdot dy = 2\cdot\int_a^b 1+\frac{\alpha\cdot\left(1+\cos^2(\varphi)\right)}{r}\cdot dy$$

This is the entire time there and back with gravity. The difference to the expected value without gravity is

$$\Delta T = 2\cdot\int_a^b \frac{\alpha\cdot\left(1+\cos^2(\varphi)\right)}{r}\cdot dy = 2\cdot\alpha\cdot\int_a^b \frac{\left(1+y^2/r^2\right)}{r}\cdot dy = 2\cdot\alpha\cdot\left[\int_a^b \frac{1}{r}\cdot dy + \int_a^b \frac{y^2}{r^3}\cdot dy\right]$$

This integral can be evaluated both numerically and analytically. For r the root of $D^2 + y^2$ is used. For a path that passes directly the solar limb, D ≈ 2.33; and using a = -499 and b = 371 the calculator delivers for the total 'delay' of the signal the value ΔT ≈ 213.3 μs ≈ 0.000,213,3 seconds - an easily measurable value.

With help of an integral table, or a computer algebra system one can find an anti-derivative:

$$\int \frac{1}{\sqrt{D^2+y^2}}\cdot dy + \int \frac{y^2}{\sqrt{D^2+y^2}^3}\cdot dy = 2\cdot\ln\left(y+\sqrt{D^2+y^2}\right) - \frac{y}{\sqrt{D^2+y^2}}$$

Setting the limits of integration and using additional symbols

a_V ~ Sun-Venus distance , y_V ~ y-coordinate of Venus , $y_V > 0$
a_E ~ Sun-Earth distance , y_E ~ y-coordinate of Earth , $y_E < 0$
φ_V ~ angle Sun-Venus-Earth
φ_E ~ angle Sun-Earth-Venus

we get the following expression for the total amount of delay which also provides good values for a path at a great distance from the sun:

$$\Delta T = 2\cdot\alpha\cdot\left[2\cdot\ln\left(\frac{a_V+y_V}{a_E+y_E}\right) + \frac{y_E}{a_E} - \frac{y_V}{a_V}\right] \approx 2\cdot\alpha\cdot\left[2\cdot\ln\left(\frac{(a_V+y_V)\cdot(a_E-y_E)}{a_E^2-y_E^2}\right) + \frac{y_E}{a_E} - \frac{y_V}{a_V}\right] \approx$$
$$\approx 2\cdot\alpha\cdot\left[2\cdot\ln\left(\frac{a_V\cdot(1+\cos(\varphi_V))\cdot a_E\cdot(1+\cos(\varphi_E))}{D^2}\right) - \cos(\varphi_E) - \cos(\varphi_V)\right]$$

The 120 foot radio antenna at MIT in Westford / USA with which Shapiro in 1966/67 carried out his first experiment.

In opposition the two angles φ_E and φ_V are very small and we may set the cosine value to 1. For this special situation this gives us the simpler formula

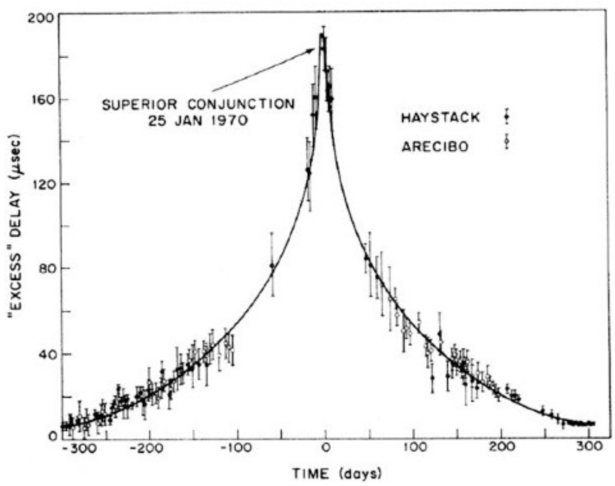
$$\Delta T \approx 2\cdot\alpha\cdot\left[2\cdot\ln\left(\frac{a_V\cdot 2\cdot a_E\cdot 2}{D^2}\right)-1-1\right] = 2\cdot\alpha\cdot\left[2\cdot\ln\left(\frac{4\cdot a_V\cdot a_E}{D^2}\right)-2\cdot\ln(e)\right] = 2\cdot\alpha\cdot\left[2\cdot\ln\left(\frac{4\cdot a_V\cdot a_E}{e\cdot D^2}\right)\right]$$

$$\text{also}\quad \Delta T \approx 4\cdot\alpha\cdot\ln\left(\frac{4\cdot a_V\cdot a_E}{e\cdot D^2}\right)$$

Even this simple formula provides for D = 2.33 (solar limb), a_E = -499 and a_V = 371 a delay of 213.3 μs.

Where are the flaws of these calculations ?
The first simplification concerns the path of integration. Light follows a geodesic and not a straight line. The angle of deviation is small (about 1.74 ") but the length of the path might be influenced. In [29-1106] Misner et al. pretend that the first ten digits are not affected from this simplification.
Second, the time delay is calculated for the observer in the OFF. So we have to correct for the STR-effect from the earth's orbital speed and for the GTR-effects of the gravitational fields of the sun and the earth (cf. problem 13 in **G6** !). Doing these calculations gives us a correction factor of 0.999'999'984'5 to adapt the time delay to earth-bound measurement. So we can ignore that, too.
However, the third flaw *has* some influence. In what type of coordinate systems do we get the distances from our astronomy programs? We did the calculations in Schwarzschild metrics. My astronomy program probably does all its calculations in Euclidean metrics in a Newtonian world. Wheeler gives in [29-1106] a simple derivation in PPN-coordinates, leading to the same formula as ours, but without Euler's e in the denominator.

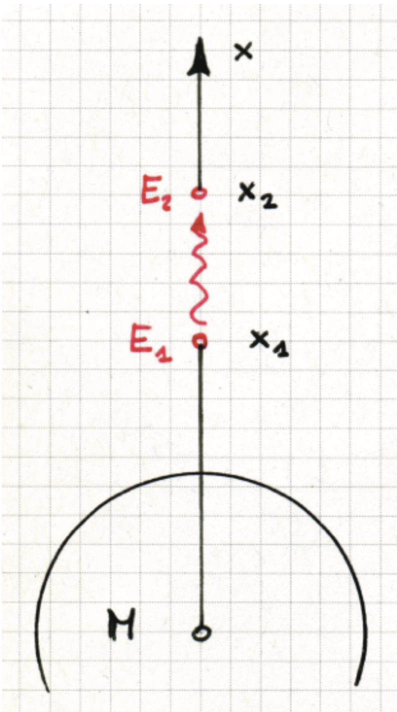
Let us therefore confront the theory with experimental data. For the Venus opposition of 1970-01-25 my astronomy program "Starry Night Pro" yields the values aE ≈ 491 , aV ≈ 363 and D ≈ 9.42 . Plugging these values into our formula we get a delay of 157 μs. Without the e in the denominator we get 177 μs, coming very close to Shapiro's measurement. However, not the delay is measured, but the total signal runtime! To get an expectation of a “zero-delay” you have to start measurements months before the day of opposition. The delay then is *calculated* from the distance Venus and Earth should have if they follow their Keplerian ellipses.



The Shapiro effect is also interesting because it decreases only slowly with increasing distance D from the central mass. The light deflection following the formula in **I2** is proportional to 1 / D . The Shapiro delay, however, is essentially proportional to 1 / ln(D), as is seen from the formula above. At a distance of 100 solar radii the value of the deflection of light decreases to 1%, but the delay there is still 21% of the maximum effect at the solar limb. One speaks, therefore, of a ‘long-range effect’.

I4 The Experiment of Rebka and Pound

We have already seen in **G4** that Schwarzschild attempted beginning in 1913 to demonstrate a ‘red-shift’ in the absorption lines of the sun’s spectrum. With this Einstein had hoped to obtain a first experimental confirmation of his theory. However, the evidence would not have specifically confirmed the GTR but rather the conservation of energy:



As a photon flies from x_1 to x_2 its potential energy increases and it therefore needs to release a bit of its internal energy $E_1 = h \cdot f_1$ (where h is Planck's constant). At location x_2 it has the smaller energy $E_2 = h \cdot f_2$, the frequency of the radiation is thus slightly smaller and the wavelength (due to the formula $c = f \cdot \lambda$) becomes somewhat bigger. The wavelength thus shifts in the direction of the red end of the optical spectrum, from whence the name ‘red-shift’ comes. For a small rise of the photon from x_1 to x_2 applies

$$\Delta E_{pot} \approx \frac{E_1}{c^2} \cdot G \cdot M \cdot \left(\frac{1}{x_2} - \frac{1}{x_1}\right) = h \cdot f_1 \cdot \frac{G \cdot M}{c^2} \cdot \left(\frac{1}{x_2} - \frac{1}{x_1}\right)$$
$$= h \cdot \alpha \cdot f_1 \cdot \left(\frac{1}{x_2} - \frac{1}{x_1}\right) = h \cdot \alpha \cdot f_1 \cdot \left(\frac{x_1 - x_2}{x_2 \cdot x_1}\right)$$

Putting this together we get

$$h \cdot (f_1 - f_2) = h \cdot \alpha \cdot f_1 \cdot \left(\frac{x_1 - x_2}{x_2 \cdot x_1}\right)$$
$$\text{oder} \quad \frac{\Delta f}{f} = \alpha \cdot \left(\frac{x_1 - x_2}{x_2 \cdot x_1}\right) = \frac{G \cdot M}{c^2} \cdot \left(\frac{x_1 - x_2}{x_2 \cdot x_1}\right)$$

For very small uplifts near the earth’s surface this can be further simplified: $x_1 \cdot x_2 \approx r_E^2$, for the difference $x_1 - x_2$ we write Δx and the term $G \cdot M / r_E^2$ is simply the gravitational acceleration g at the earth's surface. Thus we get the formula

$$\frac{\Delta f}{f} = \frac{G \cdot M}{c^2} \cdot \left(\frac{\Delta x}{r_E^2}\right) = \frac{g \cdot \Delta x}{c^2}$$

This result – albeit with a different justification – was already derived in **G4**! There we also calculated that at a height difference Δx of 22.6 meters we have a ratio $\Delta f / f$ in the range of 10^{-15} .

The two American physicists R.V. Pound and G.A. Rebka succeeded in 1960 in experimentally measuring this tiny effect with an accuracy of about 10%. In 1964 Pound and J.L. Snider increased the accuracy to 1%. They used the extremely sharp spectral lines of radioactive cobalt atoms, which due to their embedding in a crystal lattice of iron atoms are emitting and absorbing virtually recoil-free (keyword Mössbauer effect). The height difference of 22.6 meters was enough to put the source and absorber out of resonance. Then using a screw thread the absorber was moved so ‘fast’ (a few millimeters per hour, see problem 9 in **I10**) in the direction of the source until the Doppler effect again put it in resonance with the source. The speed needed for the maximum resonance was then measured to calculate the frequency shift Δf!

I5 Hafele and Keating Travel Around the World

Around 1960 the accuracy of cesium atomic clocks was so great that one could consider directly verifying the effects of STR and GTR with such clocks. While other experiments using satellites were being planned, J. Hafele and R. Keating quietly prepared a test using ordinary commercial airliners. In October 1971 they flew - first in an eastward direction and then in a westward direction - around the world. Together with the 4 atomic clocks they occupied 4 first-class seats. During the flight they continuously recorded their altitude, speed and direction.

GTR requires clocks at higher altitudes to run faster than identical clocks on the ground. STR requires that more time elapse for a clock at rest than for one moving. One must, however, be in an inertial frame which the rotating earth is not! The clocks on the ground have as a result of the earth's rotation a significant speed (given that they are not at the North Pole ...). This speed must be added to the speed of the aircraft during the eastward flight. When flying westward it must, however, be subtracted, which after extensive analysis leads to the following table of expected time differences compared to the clocks 'left behind':

Predicted Effect	Flying East	Flying West
GTR (Gravitation)	+ 144 ± 14 ns	+ 179 ± 18 ns
STR (Velocity)	- 184 ± 18 ns	+ 96 ± 18 ns
Total	- 40 ± 23 ns	+ 275 ± 21 ns

The +273 ± 7 ns measured by Hafele and Keating on the westward flight are in (almost suspiciously) good agreement with the expected value. On the eastward flight the clock with serial number 361 and its -74 ns seemed to be marching to its own drummer, but the other three with -51 to -57 ns were in harmony. The average of all four clocks was - 59 ± 10 ns.

Particularly nice aspects of this experiment are, firstly, that the effects of the STR and GTR can in a certain sense be separated, although a given clock is always measuring the total impact. Secondly, it is simply cool how these two with minimalistic resources stole the show from the others with their expensive satellite experiments.

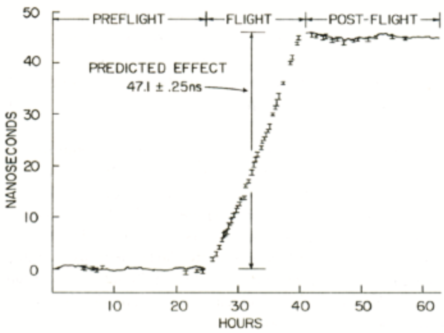
Hafele and Keating could confirm the predictions of the GTR and STR with an accuracy of about 9%. A significant increase was achieved in the experiment of Maryland, which we discuss in **I6**. With a rocket flight with a hydrogen powered maser on board which served as the clock, R.F.C. Vessot and M.W. Levine finally reached in 1979 after several years of data analysis (!!) a result with an uncertainty of ± 0.02% which is within the limits of the predictions of the STR and GTR. This rocket is now flying under the title 'Gravity Probe A' (**I8**). Today these clock experiments are performed, so to speak, in the opposite direction: We assume that the atomic clocks in the GPS satellites are 'wrong' according to STR and GTR, correct the time signals accordingly, and generate the 3D position near the earth's surface to within a few centimeters based on precise orbital data and the corrected signals from four GPS satellites (see **I7**).

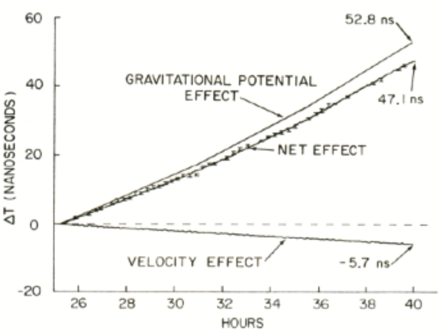
Addendum May 2009: My skepticism concerning the too-good-to-believe match between prediction and experimental results seems to be confirmed. A.G. Kelly points out in an internet post that the recently published original data of Hafele and Keating in no way allow a serious derivation of the results which made the two famous. Instead of a 'cool' or 'clever' experiment we should perhaps rather speak of fraud. The link: http://www.cartesio-episteme.net/H&KPaper.htm (september 2009)

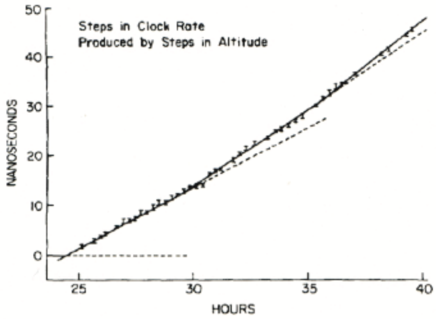
I6 The Maryland Experiment

A group of researchers from the University of Maryland (USA) carried out the Hafele and Keating experiment in a more scientific manner. Instead of ordinary commercial aircraft flying imprecise routes, they used an aircraft of the U.S. Navy whose route was very carefully tracked during the whole flight. This plane could complete its circuit in 15 hours flying at a height of up to 35,000 feet (~10,500 m). In addition, during the flight the time of the three atomic clocks on board was continually compared with that of the three clocks of the same type on the ground by exchanging laser pulses with a pulse width of 0.1 ns. The three clocks were carefully shielded against vibration, temperature variations, pressure variations and the influences of magnetic fields. The differences due to construction between the 6 clocks used in the experiment were accurately measured, before, between and after the flights, and their values were corrected according to these measurements.

After several test flights, five main flights, each lasting 15 hours were flown and analyzed. All six (corrected) clocks ran before, between and after these flights at the same speed, however, during the test flights they accumulated a difference, which exactly corresponded to the predictions of STR and GTR. The following plots are from the second of the five flights, which took place on November 22, 1975:







The second graph actually shows the same data as the first one but with an expanded time scale and also including the calculated influences of the STR and GTR. Since the flight speed (except for the relatively short periods during take-off and landing) was largely held constant (and as small as possible!), the STR 'velocity effect' shows itself as practically linear.

Including the influence of the gravitational potential one clearly sees "kinks" in the graph on the left. They arise from the fact that the aircraft had to first fly 5 hours at 25,000 feet before it had used enough fuel to allow it to fly at the next altitude of 30,000 feet. After a further 5 hours, it was light enough to climb to the targeted maximum altitude of 35,000 feet. These 'altitude levels' are particularly emphasized in the third graph.

All in all C.O. Alley and his team were able to confirm the predictions of STR and GTR to an accuracy of about 1.6%. A nice summary report of this experiment was written by Alley himself [39]. Our summary and the graphs were taken from that report.

I7 GPS, LRS and Relativity

Atomic clocks, following their introduction in the 50s, quickly became more accurate and smaller. Thus, in 2003, a rubidium atomic clock was successfully built occupying a volume of 40 cm^3, consuming 1 Watt of power and having an accuracy of $3 \cdot 10^{-12}$! Pierre Thomann of the Observatory of Neuchâtel and Gregor Dudle of the Federal Office for Metrology (METAS) in Berne also succeeded in 2003 in increasing the accuracy of the standard cesium clock using a special cooling technique by a factor of 40 to $1 \cdot 10^{-15}$.

Starting in 1958 the U.S. military began to combine these clocks with other advances in electronics and satellite technology into a worldwide global positioning system (GPS). The first working system, TRANSIT, was deployed in 1964 and primarily had the function of guiding submarine missiles to their destination. Better well-known was NAVSTAR, which was formally put into operation on July 17th, 1995 and was also available for civilian use. 24 satellites circle the earth in well-known orbits twice a day and continually transmit time signals. Small and cheap receivers can use the tiny time differences of the signals from at least 4 of these satellites to determine their own position to a few meters and also the time (four measurements determine the four unknowns). Recall: 1 nanosecond corresponds to a distance of 30 centimeters. If the receiver itself had a highly accurate and perfectly synchronized clock then two or three satellites would be sufficient to determine its position with an accuracy of a few centimeters. The uncertainty would then be mainly from the imprecise knowledge of the trajectory of the satellites.

A small group at the University of Bern is working at the forefront of this problem: In Zimmerwald laser pulses are sent to reflectors attached to the satellites specifically for this purpose. The orbit of the satellites can be precisely determined to within centimeters from the few photons of the reflected signal that can be captured. The precise orbital data allow the off-line evaluation of the GPS satellite signals with special software from the University of Bern achieving accuracy sufficient for surveying purposes. In this way one can today directly measure the folding of the Alps, the drift of continents or the earth tides. LRS (Laser Ranging Systems) are also available in Germany, in both Potsdam and Wettzell in Bavaria. All these stations are operating in a world wide collaboraton: http://ilrs.gsfc.nasa.gov/ .

The former Soviet Union has also built a military-controlled satellite navigation system (GLONASS). The European Space Agency (ESA) is preparing its own, civilian-controlled system (GALILEO). The first satellites are already in space (the launch could not be delayed because otherwise ESA would have lost a reserved frequency band ...). In addition to the ESA member states, China, India, Canada and Israel are also participating on this project. The clocks being used for the Galileo satellites have been developed by the Neuchâtel Institute mentioned above.

All of these satellite-based navigation systems would never work without taking into account the STR and GTR. The corrections for the STR, as well as those of the GTR are not even constant over an entire orbit since the orbits of the satellites are always slightly elliptical. These variations in both altitude and relative speed must be taken into account for high precision position measurements (to within a few millimeters). The influences are exactly those that we discussed in **I5** and **I6**.

Since these Global Positioning Systems have arisen in the www-age, you will find a wealth of descriptions and illustrations in the web. Also the indispensable ‘counterpart’, the LRS, is well documented in the web, although it is much less well-known to the public.

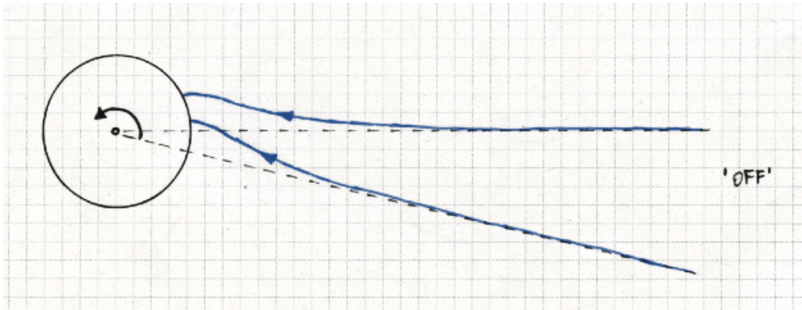
The inconspicuous facility of the Astronomical Institute of the University of Berne in Zimmerwald.

The heart of the facility at Zimmerwald: A laser which delivers ten very intense and sharp pulses per second and which are then sent to the reflector on the satellite through a small telescope (see picture above, domed building on the left). To accomplish this, the approximate position of the satellite must, of course, already be known. The few photons that arrive within the narrow time window and also have the correct wavelength will be recognized as a reflected signal and used to determine the round-trip time with a precision in the range of 0.1 nanoseconds.

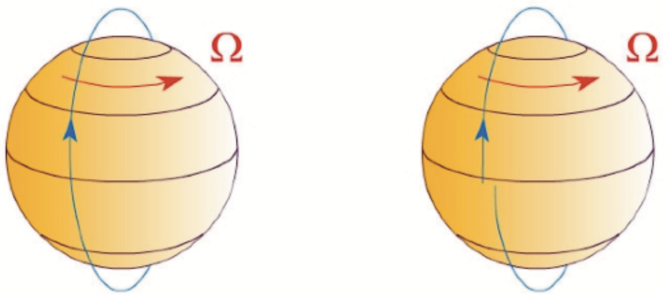
Both photos on this page are snapshots taken by the author.

I8 The Einstein-Thirring-Lense Effect

Newton's experiment with the hanging water bucket seems to show that the water in the bucket knows if it is rotating or not. Ernst Mach criticized Newton's conclusion in 1883 that absolute space exists and asserted that a rotating mass must influence an inertial frame in its vicinity. The two Austrian physicists Hans Thirring and Josef Lense showed in 1918 that Einstein's GTR solves this problem: A rotating mass slightly drags the metric of space-time along with it, twisting it a little or in the extreme case, creating a vortex-like structure in space-time. A free-falling object from the OFF no longer moves, therefore, in a 'straight' line toward the center of a spherical central mass if it is rotating:



This 'dragging' of space (keyword 'frame dragging') must be noticeable for a satellite orbiting the earth around the poles through a small rotation of its orbital plane:



according to Newton according to Einstein-Thirring-Lense

Prof. Franz Embacher has kindly granted use of this graphic from his presentation on the Lense-Thirring effect on http://homepage.univie.ac.at/Franz.Embacher/Rel/ . This website is really a true treasure!

In October 2004, I. Ciufolini and E.C. Pavlis of the University of Lecce presented their analysis [40] of the orbital data of two satellites (LAGEOS and LAGEOS 2). These satellites were specifically intended as targets for Laser Ranging (see **I7**). From the fluctuations and irregularities in their orbits geologists have gained much information on the detailed structure of earth's gravitational field as well as the density distribution in the earth's interior. Ciufolini and Pavlis have, with great effort, eliminated mathematically all other influences on the orbits of these satellites (for example, the radiation pressure of the sun!) in order to finally isolate the tiny (31 milli-arcseconds per year) Lense-Thirring effect. They believe they have succeeded to an accuracy of approximately ± 10%.

Scientists from NASA and Stanford University want to measure this effect with 1% accuracy with the satellite ‘Gravity Probe B’ using a different method. You can find detailed information on the Internet - it is fascinating how this experiment once again tests the limits of what is technically feasible! The actual measurements were completed in August 2006 and the evaluations should now (January 2007) be complete. After a critical review by experts, the outcome will be presented in April 2007. Just after the calculations of Ciufolini and Pavlis nobody expects that the GTR will be refuted. However, if Einstein’s GTR can be confirmed at the 1% level, it will mean the end of some competing theories.

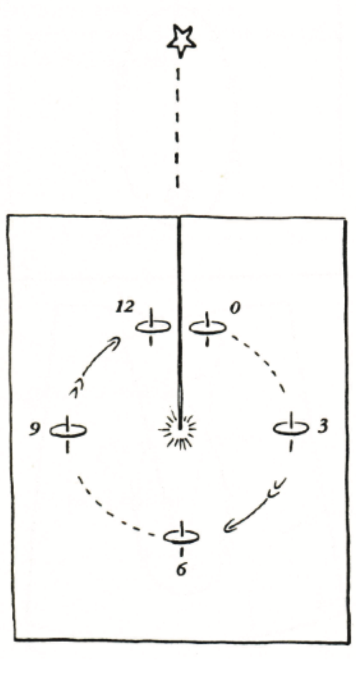
Gravity Probe B uses four gyroscopes constructed from high-precision polished quartz spheres. The changes in the gyro’s axis relative to the satellite are determined. This is aligned, using a small telescope, as stably as possible to the star HR 8703 in the constellation Pegasus, whose motion is very well known. The following two effects are simultaneously measured:

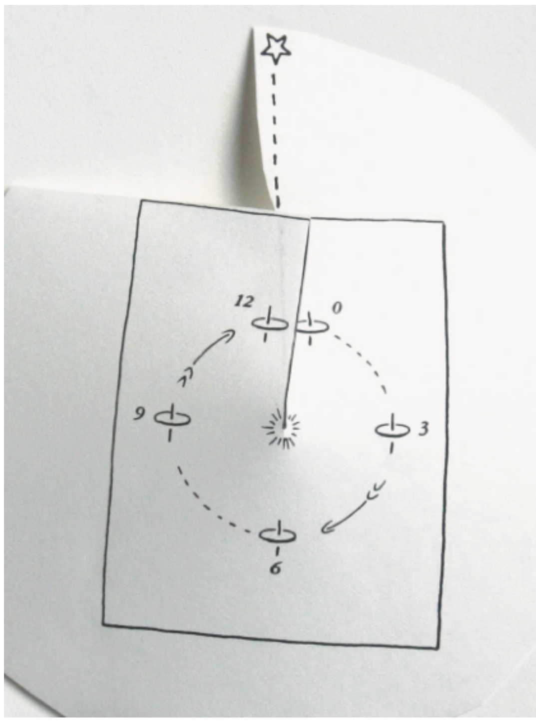
1. The ‘geodetic’ effect, which we met in **I1** as the precession of the perihelion of Mercury and consists of a tilting of the gyro axis in the orbital plane. It should be about 6.6 arc seconds. The accuracy of Gravity Probe B should be better than 0.01%.

2. The Lense-Thirring effect, which consists of a tilting of the gyro axis perpendicular to the orbital plane. It should be about 0.041 arc seconds. This effect could be measured to an accuracy of about 1%.

I would encourage you once again to have a look in the Internet at the reports, pictures and other material prepared for this latest test of the GTR. Use the search keywords mentioned above, as they are longer alive as actual concrete addresses. Addendum of January 2009: The experiment was not able to attain its lofty goal. The signal due to a ‘frame dragging effect’ is drowned in an unpredicted ‘noise’.

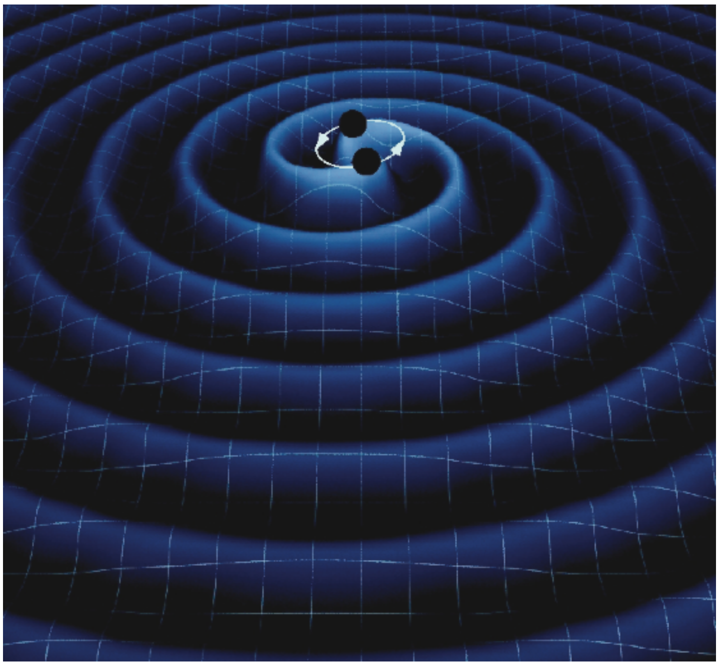
The geodetic effect can be nicely illustrated with Epstein [15-177f]. We simply repeat the idea of Section **I1** and draw an additional rotational axis. On the left we have Newton's ‘flat’ world, in which the gyro maintains its direction absolutely and on the right in the curved space of GTR the gyro is tilted through a small angle after one revolution:





I9 Gravitational Waves

The *immediate* effect of the change of a gravitational field on distant objects is a particular problem of Newton's gravitational theory! If we quickly move a mass a few meters then the corresponding effect is *immediately* felt throughout the entire Milky Way. Time does not appear at all in Newton's force law! If we move a mass periodically back and forth, it creates an immediate force at arbitrary distances, that cause a small sample mass to vibrate. Any change in a gravitational field imparts an instantaneous transmission of energy and information across arbitrary distances! However, the STR sets with the speed of light an upper limit for any information transfer.



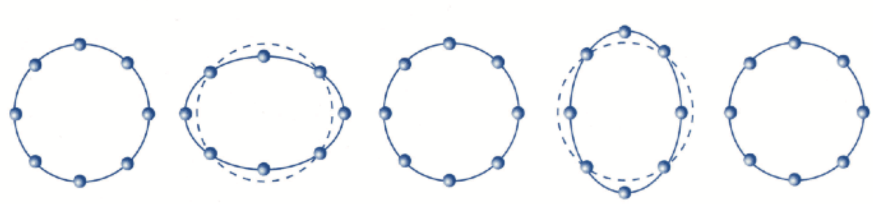
Einstein showed in 1918 that a change in the distribution of mass and therefore its impact on the structure of space-time propagates according to GTR at the speed of light. Thus two massive objects which are rapidly circling each other should emit gravitational waves which are transmitted at the speed of light throughout space.

This visualization of gravitational waves was taken from http://lisa.jpl.nasa.gov/gallery/

However, the effects on the structure of space-time, even from the most extreme sources, are extremely small. In 1987 in the Large Magellanic Cloud, a small companion galaxy of our Milky Way, a supernova explosion was observed (the explosion had taken place 160,000 years earlier ...). The ‘gravitational bolt’ on earth was a hundred times more intense than the radiation of the sun on the earth. However, it caused the distance from earth to sun to 'fluctuate' by only a few atomic diameters!

The length changes in the much shorter 'arms' of the existing detectors of such signals (300 m for TAMA300, 600 m for GEO600, 3 km for VIRGO and 4 km for LIGO) are correspondingly smaller. It has not yet been demonstrated that these facilities are capable of detecting these effects at all. Since early 2006 scientists are waiting for the first signals. One is also dependent on the cooperation of several detectors: due to the enormous background noise, one can be reasonably certain a signal has been found, only when it has simultaneously been registered by multiple, widely separated detectors.

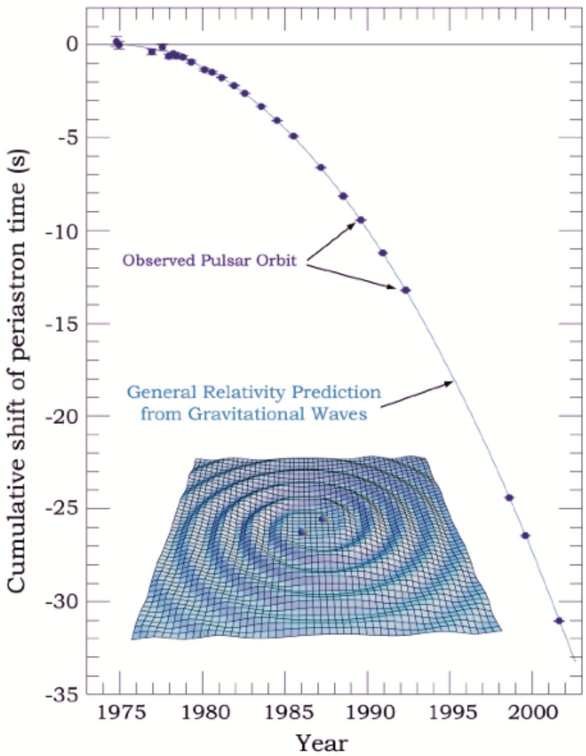
Satellite-based projects such as LISA have better chances of detecting a signal since the receiver is not exposed to terrestrial interference and since the arm length of the detectors can be tens of thousands of kilometers. More information can be found at the corresponding web-sites of NASA and ESA.



All ground based detectors have two arms perpendicular to each other. Gravitational waves are quadripole: If such a wave strikes perpendicular to the plane it affects 8 circularly arranged free falling test masses as shown above. First, the space in one direction is compressed and in the direction perpendicular to it expanded. Then the same happens in opposite directions. The metric of space-time ‘billows’ a little bit. This means that the elapsed time in the split halves of the laser beam in the two arms of the interferometer fluctuate for a short time at *fixed* mirrors. Therefore, when the two halves of the split beam are reunited the interference results no longer in permanent darkness (the default), but rather a brief flare becomes visible.

It is quite possible that tomorrow, for the first time, a gravitational wave will be clearly and directly demonstrated. This will require a bit of luck since supernova explosions, even in a large galaxy such as ours, do not occur every day. The last two were discovered by Tycho Brahe (1574) and Johannes Kepler (1604) – thus ringing the death knell on the idea of the immutability of the celestial sphere of fixed stars.

However, we have already had for a long time a very accurate indirect confirmation of Einstein’s gravitational waves. For more than 30 years, astronomers have measured the star system B1913+16. Two neutron stars of about 1.5 solar masses and a diameter of 20 km (!) orbit each other as shown in the picture above. A rotation takes less than 8 hours. Since this system radiates energy in the form of gravitational waves the two components must continually come closer. Thus, the orbital period gets shorter and shorter. This rotation can be measured very precisely, because the radio cone of one of the two stars passes the earth once each rotation: It is a pulsar. The decrease in the orbital period of 0.000,076,5 seconds per year, agrees with the prediction of GTR with an uncertainty of 0.2%. In 1993, R. Hulse and J. Taylor received the Nobel Prize in physics for the discovery and analysis of this double pulsar.



Recent findings on this topic are presented in [41]. The system PSR J0737-3039 A/B, described there, consists of two neutron stars, both of which are pulsars, and thus enable checking the predictions of GTR to an unprecedented accuracy.

I10 Problems and Suggestions

1. Check the numbers in the last column of the table of section **I1** using the values from Einstein's formula on the same page.

2. Check the numerical values claimed in the last few lines of text on the Hipparcos satellite at the end of section **I2** !

3. In the Schwarzschild metric the diameter of the circle multiplied by Pi, is greater than the corresponding circumference. Epstein's bump yields an additional precession of the perihelion in the direction in which the planet orbits (**I1**). How could you realize with paper a geometry in which the circumference is greater than the diameter times Pi? How would that impact the additional precession of the perihelion?

4. Show that the value of $\alpha = G \cdot M/c^2$ for the sun in units of 'light seconds' is about $4.9261 \cdot 10^{-6}$. What is the value of the gravitational constant G in these units?

5. We draw the cross section of the rolled up space-time as described in section **H5**. On the inside of the mass we have a spherical sector - and on the outside? For weak fields we can assume as a good approximation that y decreases there as a function of type $y = a + b / x$. Determine that solution f(x), where the x-position of the inflection point is (3 / 4)! What is the physical meaning of the x-Coordinate 3, and what is the value of R_S? Is it still a weak field? Compare clocks on the surface of the body with those in the OFF!

6. When photons rise in a gravitational field they lose energy. Will they be slower or faster for an observer in OFF?

7. How many extra seconds elapse in OFF when in Amsterdam precisely a century has elapsed? Consider the GTR (gravitation of the sun and the earth) and the STR (orbital speed of the earth).

8. Determine based on the middle diagram of section **I6** the average speed of the airplane during the 15-hour flight!

9. (Section **I4**) Determine the relative velocity v which belongs to a Doppler frequency shift of $\Delta f / f$ of $2.22 \cdot 10^{-15}$.

10. The nearly circular orbits of the NAVSTAR GPS satellites are so chosen that each satellite orbits the earth exactly twice during a sidereal day.
a) At what altitude above sea level do these satellites orbit?
b) By how many nanoseconds per orbit do the satellite clocks run fast compared to a clock at sea level at the equator, when the orbit passes over the poles?
c) Same as b) but for an orbit in the equatorial plane with the same rotational direction as the earth?
d) Same as c) but in the opposite rotation direction?

11. Visit the web-site http://homepage.univie.ac.at/Franz.Embacher/Rel/ . You can find there programs for STR and GTR. Work through some of them - you should now be well prepared to do so!

12. Search for "Michael Kramer" and "PSR J0737-3039 A/B" and read the web infos about this binary pulsar system that has turned out to be a perfect 'laboratory' to test GTR in the case of strong gravitational fields.

An Einstein poster in the background viewed through one of the high-precision quartz spheres used in the Gravity Probe B experiment as a gyroscope. It is depicted similarly to a convex lens.

http://einstein.stanford.edu/ or http://einstein.stanford.edu/gallery/